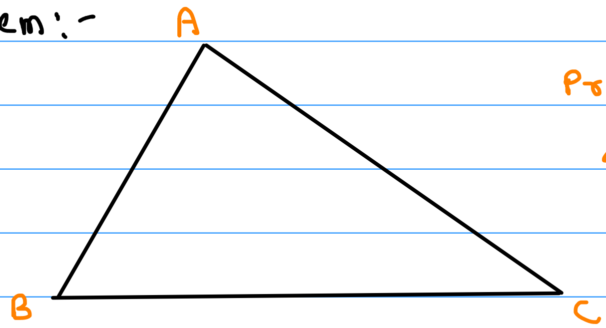
Sum of Angles

© The Philomath club

Angle sum property in polygons:-

Theorem:-

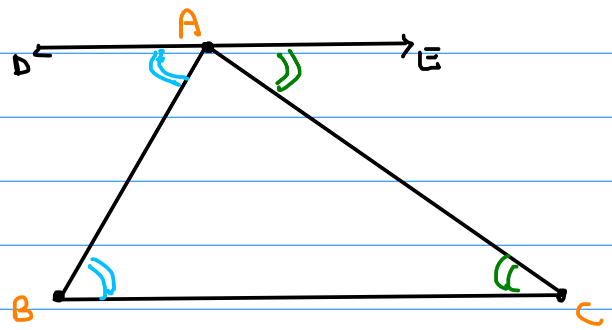


Prove that

$\angle BAC + \angle ABC + \angle BCA = 180^\circ$

Proof:-

Hint : draw parallel line passing through A, parallel to BC



$\angle DAB = \angle B$ (transversal)

$\angle EAC = \angle C$

$\angle DAB + \angle BAC + \angle EAC = 180^\circ$

$\Rightarrow \angle B + \angle A + \angle C = 180^\circ$

For quadrilateral :- 360°

Triangle $\rightarrow 180^\circ$

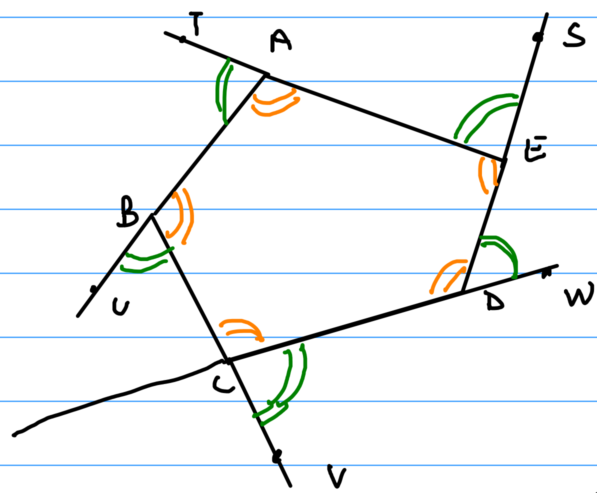
quadrilateral $\rightarrow 360^\circ$

pentagon $\rightarrow 360 + 180 = 540^\circ$

n - gon $\rightarrow (n-2)180$

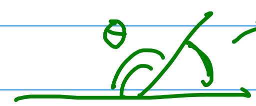
Ⓒ The Philomath club

Interior angles of a polygon :-



sum of interior angles

$\angle EAB + \angle ABC + \angle BCA + \angle CDE + \angle DEA = 540^\circ$



→ supplementary

Sum of exterior angles of ABCDE :-

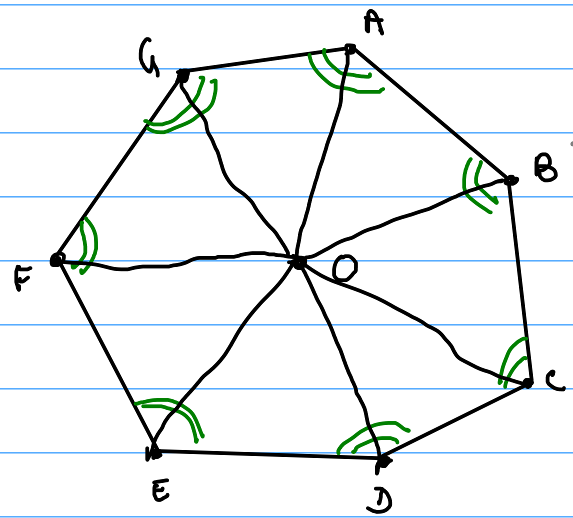
$\angle TAB + \angle UBC + \angle VCD + \angle WDE + \angle SEA = 360^\circ$

Theorem :- Sum of the interior angles of a n-gon is $(n-2) \times 180^\circ$

→ $n \times 180^\circ - 360^\circ$

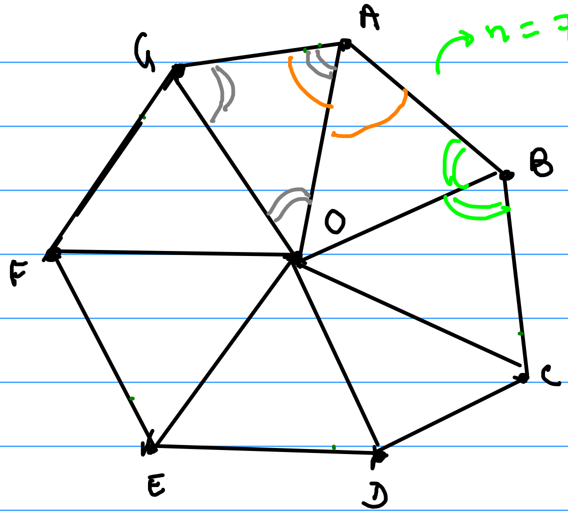
→ Proof left to the reader (For hint refer n=7 below)

→ heptagon



→ To show Sum of the interior angles

$\angle GAB + \angle ABC + \angle BCD + \angle CDE + \angle DEF + \angle EFG + \angle FGA = 900^\circ$



$n = 7$ triangles

$n \times 180^\circ = 7 \times 180^\circ$

sum of the interior angle of a triangle $= 180^\circ$

$\Rightarrow$ Sum of the interior angle of $\triangle GAO$ +
Sum of the interior angle of $\triangle AOB$ +
Sum of the interior angle of $\triangle OBC$ +
Sum of the interior angle of $\triangle OCD$ +
Sum of the interior angle of $\triangle OED$ +
Sum of the interior angle of $\triangle OEF$ +
Sum of the interior angle of $\triangle OFG$

$= \angle GAO + \angle OAG + \angle OGA$
$+$
$\angle AOB + \angle BOA + \angle BAO$
$+$
$\angle BOC + \angle OBC + \angle OCB$
$+$
$\angle OCD + \angle ODC + \angle DOC$
$+$
$\angle OED + \angle EDO + \angle DOE$
$+$
$\angle OEF + \angle EOF + \angle EFO$
$+$
$\angle OFG + \angle GOF + \angle OGF$

$= 7 \times 180^\circ$

$= 1260^\circ$

Note that :-

$\angle GAO + \angle GOA + \angle OGA$

$+$

$\angle AOB + \angle ABO + \angle BAO$

$+$

$\angle BOC + \angle OBC + \angle OCB$

$+$

$\angle OCD + \angle ODC + \angle DOC$

$+$

$\angle OED + \angle EDO + \angle DOE$

$+$

$\angle OEF + \angle EOF + \angle EFO$

$+$

$\angle OFG + \angle GOF + \angle OGF$

magic!!

Note that

$\angle GAO + \angle BAO = \angle GAB$

$\angle ABO + \angle OBC = \angle ABC$

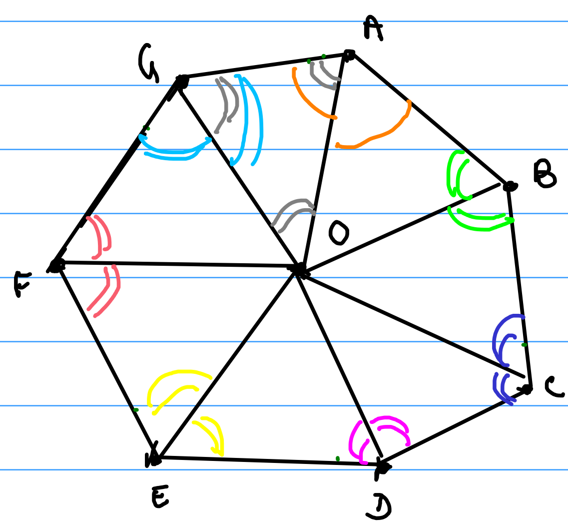
$\angle OCB + \angle OCD = \angle BCD$

$\angle ODC + \angle EDO = \angle CDE$

$\angle OED + \angle OEF = \angle DEF$

$\angle OFG + \angle EFO = \angle EFG$

$\angle OGA + \angle OGF = \angle FGA$

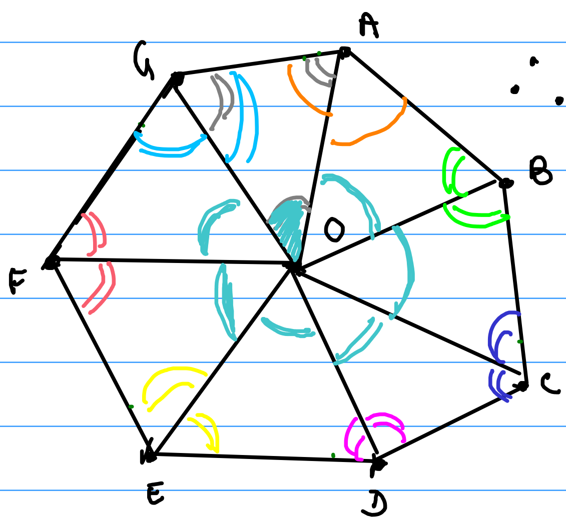


$$7\times 180^\circ = 1260^\circ$$

$$\parallel$$

$$\begin{aligned}
\therefore\ & \angle GAO + \angle OAG + \angle OGA \\
+\ & \angle AOB + \angle BOA + \angle BAO \\
+\ & \angle BOC + \angle OBC + \angle OCB \\
+\ & \angle OCD + \angle ODC + \angle DOC \\
+\ & \angle OED + \angle EDO + \angle DOE \\
+\ & \angle OEF + \angle EOF + \angle EFO \\
+\ & \angle OFG + \angle GOF + \angle OGF
\end{aligned}
= \begin{aligned}
& \angle GAB + \angle ABC + \angle BCD \\
& + \angle CDE + \angle DEF \\
& + \angle EFG + \angle FGA \\
& + \angle GOA + \angle AOB \\
& + \angle BOC \\
& + \angle COD + \angle DOE \\
& + \angle EOF + \angle GOF
\end{aligned}$$

Full rotation / circle = 360°



$$\begin{aligned}
\therefore\ & \angle GAB + \angle ABC + \angle BCD \\
& + \angle CDE + \angle DEF \\
& + \angle EFG + \angle FGA = 7\times 180^\circ \\
& + \angle GOA + \angle AOB \\
& + \angle BOC \\
& + \angle COD + \angle DOE \\
& + \angle EOF + \angle GOF
\end{aligned}$$

But
$$\begin{aligned}
& \angle GOA + \angle AOB \\
& + \angle BOC \\
& + \angle COD + \angle DOE \\
& + \angle EOF + \angle GOF
\end{aligned} = 360^\circ = 2\times 180^\circ$$

So

$$\angle GAB + \angle ABC + \angle BCD + \angle CDE + \angle DEF + \angle EFG + \angle FGA = 7 \times 180^\circ - 2 \times 180^\circ$$

$$= (7-2) \times 180^\circ$$

$$= 5 \times 180^\circ$$

$$= 900^\circ$$

Hint for any n:

① Take point O inside the n-gon

② We have n triangles

③ Sum of the interior angle of the n triangle $= n \times 180^\circ$ →

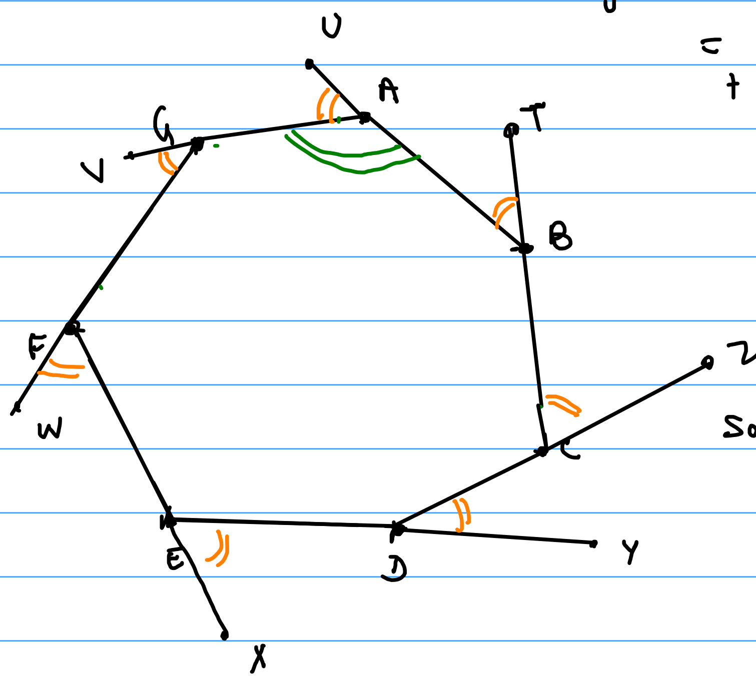
④ Do the "magic trick"

⑤ Subtract 360° (for center angles) → $2 \times 180^\circ$

$$n \times 180^\circ - 2 \times 180^\circ = (n-2) \times 180^\circ$$

Corollary : Sum of the exterior angles of a n-gon is $2 \times 180^\circ$

Proof :-



Sum of the exterior angles

$= \angle UAG + \angle VGF + \angle WFE$
$+ \angle XED + \angle YDC + \angle ZCB$
$+ TBA$

Note that $\angle UAG = 180 - \angle GAB$
(Because UAB is a straight line)

So,

$\angle VGF = 180 - \angle FGA$
$\angle WFE = 180 - \angle EFG$
$\angle XED = 180 - \angle DEF$
$\angle CDY = 180 - \angle CDE$
$\angle ZCB = 180 - \angle BCD$
$\angle TBA = 180 - \angle ABC$
$\angle UAG = 180 - \angle GAB$

$\therefore \angle UAG + \angle VGF + \angle WFE$
$+ \angle XED + \angle YDC + \angle ZCB$
$+ TBA$ $= 180 - \angle FGA$
$+ 180 - \angle EFG$
$+ 180 - \angle DEF$
$+ 180 - \angle CDE$
$+ 180 - \angle BCD$
$+ 180 - \angle ABC$
$+ 180 - \angle GAB$

$= 180 \times 7 - (\angle FGA + \angle EFG + \angle DEF + \angle CDE$
$+ \angle BCD + \angle ABC + \angle GAB)$

$= 180^\circ \times 7 - (180^\circ \times (7-2))$

$= 2 \times 180^\circ$

Hint :- → For n gon

1. supplementary

2. $n \times 180^\circ - (\text{sum of interior angles})$

$= n \times 180^\circ - (n-2) \times 180^\circ$

$= 2 \times 180^\circ$